Homework 3
MTH 829 Complex Analysis

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Proposition 0.1 (Exercise IV.8.1). *The solution set to* $\cos z = 2$ *is*

$$\{x + iy \in \mathbb{C} : x = 2\pi n, y = -\log(2 \pm \sqrt{3}), n \in \mathbb{Z}\}$$

Proof. First we reduce to a quadratic in e^{iz}.

$$2 = \cos z = \frac{e^{iz} + e^{-iz}}{2} \implies e^{iz} + e^{-iz} - 4 = 0 \implies \left(e^{iz}\right)^2 - 4e^{iz} + 1 = 0$$

Then applying the quadratic formula,

$$e^{iz} = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$$

Let $z = x + iy$, then

$$e^{iz} = e^{i(x+iy)} = e^{-y+ix} = e^{-y}(\cos x + i \sin x)$$

Since e^{iz} is real, $\sin x = 0$, so x is an integer multiple of π. Thus $\cos x = \pm 1$. Since $2 + \sqrt{3}$ and $2 - \sqrt{3}$ are positive, we must have $\cos x = +1$. Thus x is an integer multiple of 2π.

$$e^{iz} = 2 + \sqrt{3} \implies e^{-y} = 2 + \sqrt{3} \implies -y = \log(2 + \sqrt{3})$$
$$e^{iz} = 2 - \sqrt{3} \implies e^{-y} = 2 - \sqrt{3} \implies -y = \log(2 - \sqrt{3})$$

Thus the solution set is

$$\{x + iy : x = 2\pi n, y = -\log(2 \pm \sqrt{3}), n \in \mathbb{Z}\}$$

□

Lemma 0.2 (Exercise IV.8.2, not assigned). *Let* $z = x + iy$. *Then*

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$
$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

Proposition 0.3 (Exercise IV.8.3)**.** *Let ϕ be the map $z \mapsto \cos z$. Let $c \in \mathbb{C}$. The image of a line* $\operatorname{Re} z = c$ *is a hyperbola with transverse axis in the real direction, except for degenerate cases when* $\sin c = 0$ *or* $\cos c = 0$. *The image of a line* $\operatorname{Im} z = c$ *is an ellipse centered at the origin, except for the degenerate case $c = 0$.*

Proof. Fix $c \in \mathbb{R}$. First we determine the image of the line $\operatorname{Im} z = c$ under $\cos z$. When $c = 0$, we take the image of the embedded real axis, which is the line segment $[-1, 1]$. When $c \neq 0$, note that $\sinh c$ and $\cosh c$ are nonzero, so using the formula

$$\cos(x + ic) = u + iv = \cos x \cosh c - i \sin x \sinh c$$

We notice that the real and imaginary parts u, v satisfy

$$\frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = \left(\frac{\cos x \cosh c}{\cosh c}\right)^2 + \left(\frac{-\sin x \sinh c}{\sinh c}\right)^2 = \cos^2 x + \sin^2 x = 1$$

so the image is an ellipse centered at the origin. It intersects the real axis at $\pm \cosh c$, and intersects the imaginary axis at $\pm \sinh c$. Now we determine the image of the line $\operatorname{Re} z = c$. Applying the formula again,

$$\cos(c + iy) = u + iv = \cos c \cosh y - i \sin c \sinh y$$

When $\sin c = 0$, the image is contained in the real axis, and when $\cos c = 0$, the image is contained in the imaginary axis. When $\sin c \neq 0$ and $\cos c \neq 0$, we see that

$$\frac{u^2}{\cos^2 c} - \frac{v^2}{\sin^2 c} = \left(\frac{\cos c \cosh y}{\cos c}\right)^2 - \left(\frac{\sin c \sinh y}{\sin c}\right)^2 = \cosh^2 y - \sinh^2 y = 1$$

so the image is a hyperbola centered at the origin with the real axis as the transverse axis. It intersects the real axis at $\pm \cos c$. $\square$

Proposition 0.4 (Exercise IV.13.3)**.** *Let G be the open set $\mathbb{C} \setminus [-1, 1]$. There is a branch of $\sqrt{\frac{z+1}{z-1}}$ in G.*

Proof. Let $\phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be given by $z \mapsto \frac{z+1}{z-1}$. Then $\phi(1) = \infty$, $\phi(0) = -1$, and $\phi(-1) = 0$, so ϕ maps the extended real line onto itself (because ϕ is a fractional linear transformation and preserves clircles). In particular ϕ maps the segment $[-1, 1]$ onto the negative real axis $(-\infty, 0]$. We know that there is branch of log on the slit complex plane $\mathbb{C} \setminus (-\infty, 0]$, so there is a branch of log on $\phi(G)$. Then by the discussion in IV.13, there is a branch of $\phi^{1/2}$ in G. $\square$

For exercises V.6.2 and following, I need the following lemma, which is Exercise V.6.1, which was not assigned. I won't mentioned the use of this lemma, I'll just use the phrases "converges uniformly" and "uniformly Cauchy" interchangeably.

Lemma 0.5 (Exercise V.6.1, not assigned)**.** *Let g_n be a sequence of complex-valued functions. The sequence converges uniformly on G if and only if it is uniformly Cauchy on G.*

Proof. First assume that g_n converges uniformly to g on G. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ so that for all $z \in G$,

$$n \geq N \implies |g_n(z) - g(z)| < \epsilon/2$$

Then if $n, m \geq N$, we have both $|g_n(z) - g(z)| < \epsilon/2$ and $|g_m(z) - g(z)| < \epsilon/2$, so by the triangle inequality,

$$n, m \geq N \implies |g_n(z) - g_m(z)| \leq |g_n(z) - g(z)| + |g_m(z) - g(z)| < \epsilon$$

so the sequence is uniformly Cauchy in G. Now suppose that g_n is uniformly Cauchy in G. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ so that for $z \in G$,

$$n, m \geq N \implies |g_n(z) - g_m(z)| < \epsilon/2$$

For $z \in G$, define $g(z) = \lim_{n\to\infty} g_n(z)$. This limit always exists because $g_n(z)$ is a Cauchy sequence for a fixed z. Now choose $m \geq N$. Then

$$\begin{aligned} n \geq N &\implies |g_n(z) - g_m(z)| < \epsilon \\ &\implies g_m(z) - \epsilon < g_n(z) < g_m(z) + \epsilon \\ &\implies g_m(z) - \epsilon/2 \leq \lim_{n\to\infty} g_n(z) \leq g_m(z) + \epsilon/2 \\ &\implies g_m(z) - \epsilon < g_(z) < g_m(z) + \epsilon \\ &\implies |g_m(z) - g(z)| < \epsilon \end{aligned}$$

Since m was arbitrary, this says that g_n converges uniformly to g. □

Lemma 0.6 (for Exercise V.6.2)**.** *Let g_n be a sequence of complex valued functions defined on an open set $G \subset \mathbb{C}$. Suppose that g_n converges uniformly on open sets $U_1, \ldots, U_k$. Then g_n converges uniformly on $\bigcup_k U_k$.*

Proof. First consider the case of just two open sets U_1, U_2. Let $\epsilon > 0$. By hypothesis, there exists $N_1, N_2 \in \mathbb{N}$ so that

$$\begin{aligned} n, m \geq N_1 &\implies |g_n(z) - g_m(z)| < \epsilon, \quad \forall z \in U_1 \\ n, m \geq N_2 &\implies |g_n(z) - g_m(z)| < \epsilon, \quad \forall z \in U_2 \end{aligned}$$

Let $N = \max(N_1, N_2)$. Then for $z \in U_1 \cup U_2$, we have $z \in U_1$ or $z \in U_2$, so

$$n, m \geq N \implies |g_n(z) - g_m(z)| < \epsilon$$

By a straightforward induction, the result for k open sets follows. □

Proposition 0.7 (Exercise V.6.2)**.** *Let g_n be a sequence of complex valued functions defined in an open set $G \subset \mathbb{C}$. It converges locally uniformly in G if and only if it converges uniformly on each compact subset of G.*

Proof. First suppose that g_n converges locally uniformly in G. Let A be a compact subset of G. By hypothesis, for each $z \in G$, there is an open neighborhood U_z on which g_n converges uniformly. The collection $\{U_z : z \in G\}$ is an open cover of A. Since A is compact, there is a finite subcover $\{U_{z_k}\}_{k=1}^N$. By the previous lemma, g_n converges uniformly on $\bigcup_k U_{z_k}$, which contains A. Thus g_n converges uniformly on A.

Now suppose that g_n converges uniformly on each compact subset of G. By openness of G, there exists $r > 0$ so that $B_r(z) \subset G$. Then $\overline{B}_{r/2}(z) \subset B_r(z) \subset G$. Since $\overline{B}_{r/2}(z)$ is compact, g_n converges uniformly on it by hypothesis. Then g_n converges uniformly on $B_{r/2}(z)$, which is an open neighborhood of z. Hence g_n converges locally uniformly in G. □

Proposition 0.8 (Exercise V.7.1)**.** *Suppose that* $\sum_{n=0}^\infty |f_n|$ *converges locally uniformly in* G*. Then* $\sum_{n=0}^\infty f_n$ *converges locally uniformly in* G*.*

Proof. Let $z \in G$ and let $\epsilon > 0$. By hypothesis, there exists a neighborhood U of z on which $\sum_{n=0}^\infty |f_n|$ converges uniformly. That is, there exists $N \in \mathbb{N}$ so that $\forall z \in U$ we have

$$k \geq m \geq N \implies \left| \sum_{n=0}^{k} |f_n(z)| - \sum_{n=0}^{m} |f_n(z)| \right| < \epsilon$$

We claim that $\sum_{n=0}^\infty f_n$ converges uniformly on U as well. Assuming $k \geq m$,

$$\begin{aligned} \left| \sum_{n=0}^{k} f_n(z) - \sum_{n=0}^{m} f_n(z) \right| &= \left| \sum_{n=0}^{k-m} f_n(z) \right| \leq \sum_{n=0}^{k-m} |f_n(z)| \\ &= \sum_{n=0}^{k} |f_n(z)| - \sum_{n=0}^{m} |f_n(z)| \leq \left| \sum_{n=0}^{k} |f_n(z)| - \sum_{n=0}^{m} |f_n(z)| \right| \end{aligned}$$

Thus

$$k \geq m \geq N \implies \left| \sum_{n=0}^{k} f_n(z) - \sum_{n=0}^{m} f_n(z) \right| \leq \left| \sum_{n=0}^{k} |f_n(z)| - \sum_{n=0}^{m} |f_n(z)| \right| < \epsilon$$

□

Proposition 0.9 (Exercise V.7.2)**.** *The series* $\sum_{n=0}^\infty \left(\frac{z-1}{z+1}\right)^n$ *converges locally uniformly in the half-plane* $\operatorname{Re} z > 0$*. The sum is* $\frac{z+1}{2}$*.*

Proof. The series is undefined for $z = -1$, so we assume $z \neq -1$. By the result in V.7, a geometric series $\sum_{n=0}^\infty w^n$ converges locally uniformly to $\frac{1}{1-w}$ on $|w| < 1$. Thus our series with $w = \frac{z-1}{z+1}$ converges locally uniformly on

$$\begin{aligned} \left| \frac{z-1}{z+1} \right| < 1 &\iff |z-1| < |z+1| \iff |x-1+iy| < |x+1+iy| \\ &\iff \sqrt{(x-1)^2+y^2} < \sqrt{(x+1)^2+y^2} \iff (x-1)^2 < (x+1)^2 \\ &\iff x^2-2x+1 < x^2+2x+1 \iff -2x < 2x \iff x > 0 \end{aligned}$$

That is, the series converges locally uniformly for $\operatorname{Re} z > 0$, and the sum is

$$\frac{1}{1 - \frac{z-1}{z+1}} = \frac{z+1}{(z+1)-(z-1)} = \frac{z+1}{2}$$

□

Lemma 0.10 (for Exercise V.12.2)**.** *Let x_n be a sequence of non-negative real numbers. Then*

$$\limsup_{n\to\infty} (x_n)^{n/(n-1)} = \limsup_{n\to\infty} x_n$$

Proof. Let exp and log denote the real exponential and natural logarithm functions. Using limit laws including the fact that log commutes with limits and products commute with limits,

$$\begin{aligned}\limsup_{n\to\infty}(x_n)^{n/(n-1)} &= \exp\log\limsup_{n\to\infty}(x_n)^{n/(n-1)}) = \exp\limsup_{n\to\infty}\log((x_n)^{n/(n-1)}\\ &= \exp\limsup_{n\to\infty}\left(\left(\frac{n}{n-1}\right)\log x_n\right) = \exp\left(\left(\limsup_{n\to\infty}\frac{n}{n-1}\right)\left(\limsup_{n\to\infty}\log x_n\right)\right)\\ &= \exp\limsup_{n\to\infty}\log x_n = \exp\log\limsup_{n\to\infty} x_n = \limsup_{n\to\infty} x_n\end{aligned}$$

□

Proposition 0.11 (Exercise V.12.2)**.** *A power series $\sum_{n=0}^{\infty} a_n z^n$ and its termwise derivative $\sum_{n=1}^{\infty} n a_n z^{n-1}$ have the same radius of convergence.*

Proof. By the Cauchy-Hadamard theorem, the radius R of convergence for the original power series is

$$R = \frac{1}{\limsup_{n\to\infty}|a_n|^{1/n}}$$

and the radius $\widetilde{R}$ of convergence for the termwise derivative is

$$\widetilde{R} = \frac{1}{\limsup_{n\to\infty}|na_n|^{1/(n-1)}}$$

We claim that these two lim sup expressions are equal. We know that

$$\lim_{n\to\infty} n^{1/(n-1)} = 1$$

so we have

$$\limsup_{n\to\infty}|na_n|^{1/(n-1} = \limsup_{n\to\infty} n^{1/(n-1)}\limsup_{n\to\infty}|a_n|^{1/(n-1)} = \limsup_{n\to\infty}|a_n|^{1/(n-1)}$$

Then by the above lemma we have

$$\limsup_{n\to\infty}|a_n|^{1/(n-1)} = \limsup_{n\to\infty}\left(|a_n|^{1/n}\right)^{n/(n-1)} = \limsup_{n\to\infty}|a_n|^{1/n}$$

Thus the two lim sup expressions are equal, so $R = \widetilde{R}$. □

Lemma 0.12 (for Exericse V.14.1c)**.**

$$\lim_{k\to\infty} \left(\frac{1}{k}\right)^{1/k!} = 1$$

Proof. First, we apply the exponential and logarithm to reduce the problem to computing the limit inside the exponential.

$$\begin{aligned}\lim_{k\to\infty} \left(\frac{1}{k}\right)^{1/k!} &= \exp\log\lim_{k\to\infty} \left(\frac{1}{k}\right)^{1/k!} = \exp\lim_{k\to\infty}\log\left(\frac{1}{k}\right)^{1/k!} \\ &= \exp\lim_{k\to\infty}\left(\frac{1}{k!}\log\frac{1}{k}\right) = \exp\left(-\lim_{k\to\infty}\frac{\log k}{k!}\right)\end{aligned}$$

Next observe that

$$\log k < k \implies \frac{\log k}{k!} < \frac{k}{k!} = \frac{1}{(k-1)!} \implies \lim_{k\to\infty}\frac{\log k}{k!} \leq \lim_{k\to\infty}\frac{1}{(k-1)!} = 0$$

Since the terms of $\frac{\log k}{k}$ are positive, the limit must be precisely zero. Thus our original limit is

$$\lim_{k\to\infty} \left(\frac{1}{k}\right)^{1/k!} = \exp(-0) = 1$$

□

Lemma 0.13 (for Exercise V.14.1d)**.**

$$\lim_{n\to\infty} (n!)^{1/n} = \infty$$

Proof. First we establish an inequality.

$$(n!)^2 = (n)(n-1)\ldots,(1)(n)(n-1)\ldots(1) = \big[(n)(1)\big]\big[(n-1)(2)\big]\ldots\big[(2)(n-1)\big]\big[(n)(1)\big]$$

Each term $(k+1)(n-k)$ is greater than n. Therefore

$$(n!)^2 \geq n^n \implies (n!)^(1/n) \geq \sqrt{n}$$

Thus

$$\lim_{n\to\infty}(n!)^(1/n) \geq \lim_{n\to\infty}\sqrt{n} = \infty$$

□

(Exercise V.14.1)

(a) We apply the ratio test to find the radius of convergence for $\sum_{n=0}^{\infty}\frac{z^n}{n^3}$. The following limit exists, so $R = 1$.

$$\lim_{n\to\infty}\left|\frac{n^3}{(n+1)^3}\right| = 1$$

(c) Consider the series $\sum_{n=1}^{\infty} \frac{z^{n!}}{n}$. We can rewrite it as

$$\sum_{n=1}^{\infty} \frac{z^{n!}}{n} = z + \frac{z^2}{2} + 0z^3 + 0z^4 + 0z^5 + \frac{z^6}{3} + \ldots = \sum_{n=1}^{\infty} a_n z^n$$

where

$$a_n = \begin{cases} \frac{1}{k} & \exists k \in \mathbb{N}, n = k! \\ 0 & \forall k \in \mathbb{N}, n \neq k! \end{cases}$$

Then

$$|a_n|^{1/n} = \begin{cases} \left(\frac{1}{k}\right)^{(1/k!)} & \exists k \in \mathbb{N}, n = k! \\ 0 & \forall k \in \mathbb{N}, n \neq k! \end{cases}$$

Then using the above lemma we can compute

$$\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{k\to\infty} \left(\frac{1}{k}\right)^{(1/k!)} = 1$$

Then by the Cauchy-Hadamard theorem, the radius of convergence is the reciprocal of this limsup, so $R = 1$.

(d) Consider the series $\sum_{n=0}^{\infty} (n!) z^{n!}$. Then we have

$$a_n = \begin{cases} n! & \exists k \in \mathbb{N}, n = k! \\ 0 & \forall k \in \mathbb{N}, n \neq k! \end{cases} \implies |a_n|^{1/n} = \begin{cases} n!^{1/n} & \exists k \in \mathbb{N}, n = k! \\ 0 & \forall k \in \mathbb{N}, n \neq k! \end{cases}$$

Using the calculation from a previous lemma,

$$\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} (n!)^{1/n} = \infty$$

so the radius of convergence is zero.

(e) Consider the series $\sum_{n=1}^{\infty} n^n z^{n^2}$. Then aft9er rewriting it to include zero terms,

$$\sum_{n=1}^{\infty} n^n z^{n^2} = 1^1 z^1 + 0z^2 + 0z^3 + 2^2 z^{2^2} + 0z^5 + \ldots + 3^3 z^{3^2} + \ldots$$

we can formulate the nth term of this series as

$$a_n = \begin{cases} k^k & \exists k \in \mathbb{N}, k^2 = n \\ 0 & \forall k \in \mathbb{N}, k^2 \neq n \end{cases} = \begin{cases} \sqrt{n}^{\sqrt{n}} & \exists k \in \mathbb{N}, k^2 = n \\ 0 & \forall k \in \mathbb{N}, k^2 \neq n \end{cases}$$

Then we take the lim sup

$$\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} \left(\sqrt{n}^{\sqrt{n}}\right)^{1/n} = \limsup_{n\to\infty} n^{1/(2\sqrt{n})}$$

We can compute this as a limit:

$$\lim_{n\to\infty} n^{1/(2\sqrt{n})} = \exp \lim_{n\to\infty} \log n^{1/(2\sqrt{n})} = \exp \lim_{n\to\infty} \frac{\log n}{2\sqrt{n}}$$
$$= \exp \lim_{n\to\infty} \frac{n^{-1}}{n^{-1/2}} = \exp \lim_{n\to\infty} n^{-1/2} = \exp 0 = 1$$

Thus the radius of convergence is 1.

Proposition 0.14 (Exercise V.16.2)**.** *The power series*

$$\sum_{n=1}^{\infty} n^2 z^n$$

represents the function

$$f(z) = \frac{-z(z+1)}{(z-1)^3}$$

on the unit disk $|z| < 1$.

Proof. We know that the series $\sum_{n=0}^{\infty} z^n$ represents the function $z \mapsto \frac{1}{1-z}$ on $|z| < 1$. Taking the termwise derivative tells us that $\frac{\partial}{\partial z}\left(\frac{1}{1-z}\right) = \frac{1}{(1-z)^2}$ is represented by $\sum_{n=1}^{\infty} nz^{n-1}$ also on $|z| < 1$. Multiplying by z on both sides of the "equality," $\frac{z}{(1-z)^2}$ is represented by $\sum_{n=1}^{\infty} nz^n$ on $|z| < 1$. Taking another termwise derivative and then multiplying by z again gives that

$$z\frac{\partial}{\partial z}\left(\frac{z}{(1-z)^2}\right) = \frac{-z(z+1)}{(z-1)^3}$$

is represented by $\sum_{n=1}^{\infty} n^2 z^n$ on $|z| < 1$. □

Proposition 0.15 (Exercise V.16.3, part one)**.** *Let k be a nonnegative integer. Then the series*

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{k+2n}}{n!(n+k)!}$$

has radius of convergence ∞.

Proof. Let b_n be the nth term of the above series. Then

$$\left|\frac{b_{n+1}}{b_n}\right| = \frac{\left(\frac{\left(\frac{z}{2}\right)^{k+2n+2}}{(n+1)!(n+1+k)!}\right)}{\left(\frac{\left(\frac{z}{2}\right)^{k+2n}}{n!(n+k)!}\right)} = \frac{\left(\frac{z}{2}\right)^2}{(n+1)(n+1+k)}$$

so

$$\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = 0$$

for al z, k. Thus by the ratio test, the series converges absolutely for all z, so the radius of convergence is infinite. □